

ON THE KUMMER CONSTRUCTION

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ABSTRACT. We discuss a generalization of the Kummer construction. Namely an integral representation of a finite group produces an action on an abelian variety and, via a crepant resolution of the quotient, this gives rise to a higher dimensional variety with trivial canonical class and first cohomology. We use virtual Poincaré polynomials with coefficients in a ring of representations and McKay correspondence to compute cohomology of such Kummer varieties.

1. INTRODUCTION

Kummer surfaces are constructed in a two step process: (1) divide an abelian surface by an action of an involution, (2) resolve singularities of the quotient, which arise from the fixed points of the action, by blowing them up to (-2) -curves. The result of this process is a K3 surface, this is because the group action kills the fundamental group of the abelian surface and preserves the canonical form, and also because the resolution is crepant. The invariants of this surface can be computed by looking at the invariants of the involution and the contribution of the resolution. This construction is classical, see [Kum75] or [BPVdV84].

It is natural to ask about a generalization of the above procedure. This involves dividing an abelian variety by an action of a finite group. Our set up is as follows:

- G is a finite group with an irreducible integral representation $\rho_{\mathbb{Z}} : G \rightarrow GL(r, \mathbb{Z})$ whose fixed point set is $\{0\}$,
- A is a complex abelian variety of dimension d , with neutral element, addition and subtraction denoted by $0, \pm$; note that the construction can be carried over starting with a compact complex torus as well. If d is odd we assume additionally $\det(\rho_{\mathbb{Z}}) = 1$, that is $\rho_{\mathbb{Z}} : G \rightarrow SL(r, \mathbb{Z})$.

The first step of the generalized Kummer construction is achieved by the induced action

$$(1.1) \quad \rho_A = \rho_{\mathbb{Z}} \otimes_{\mathbb{Z}} A : G \longrightarrow \text{Aut}(A^r)$$

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which is obtained by identification $A^r = \mathbb{Z}^r \otimes_{\mathbb{Z}} A$. In other words, G acts on A^r with integral matrices coming from the representation $\rho_{\mathbb{Z}}$ and we consider the quotient $Y := A^r/G$.

If a crepant resolution of Y exists, $X \rightarrow Y$, we obtain a manifold X of dimension rd with $K_X \sim 0$ and $H^1(X, \mathbb{C}) = 0$, as in the original Kummer construction. The former comes from our assumptions regarding both the action and the resolution. The latter follows because $H^1(A^r/G, \mathbb{C}) = H^1(A^r, \mathbb{C})^G = (\mathbb{C}^{2rd})^G$ is trivial by our assumptions and the crepant resolution of A^r/G does not change first cohomology, as one can see using for instance the Leray spectral sequence and the rationality of quotient singularities.

Computing higher cohomology of X in general is a hard task and we need the following extra properties of the (crepant) resolution. Let \mathfrak{a} be the tangent space of A at identity (or at any point p), i.e. the Lie algebra of holomorphic vector fields tangent to A . The induced action $\rho_{\mathfrak{a}} = \rho_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathfrak{a} : G \rightarrow GL(\mathfrak{a}^r)$, which splits into d copies of complexified representation $\rho_{\mathbb{C}} = \rho_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$, is the tangent action. For any point $p \in A^r$ with non trivial isotropy group G_p the action $\rho_{\mathfrak{a}|G_p} = d \cdot \rho_{\mathbb{C}|G_p}$ is a representation of $T_p A^r \simeq \mathbb{C}^{dr}$. In analytic or étale topology the action of G_p around p is equivalent to the action $\rho_{\mathfrak{a}|G_p}$ in a neighborhood of 0 in $T_p A^r \simeq \mathbb{C}^{dr}$. For more details on the notation see the next section.

We will assume that:

- a) Over the set of points of A^r/G which represent orbits with the same isotropy the resolution is a locally product (see the definition 3.4).
- b) McKay correspondence holds for the crepant resolutions of a quotient singularity \mathbb{C}^{rd}/G_p ; that is there exists a canonical relation between the conjugacy classes of elements in $H = G_p$ and the cohomology of the crepant resolution (see e.g. [Rei02], [BM94], [GK04]).

By standard group action arguments we will have a description of the action of G on the set of points with the same isotropy (see 3.5); then we will glue these local resolutions to a global projective resolution $X \rightarrow Y = A^r/G$. This step may be non-obvious in case when the singular points are non-isolated and their resolution is not obtained in a canonical way; here we impose the assumption that the resolution is a locally product. Under the above assumptions, at the end of this procedure, we obtain a general formula to compute the Poincaré polynomial $P_X(t)$ of X , that is the Betti numbers of X , as summarized in 3.9.

In the second part of the paper we consider some examples which satisfy our assumptions. In particular in section 4 we start with $d = 1$, i.e. A is an elliptic curve, and we apply the construction to some representations in $SL(2, \mathbb{Z})$ and in $SL(3, \mathbb{Z})$. We compute in these cases, with our procedure, the cohomology of the resulting Kummer surfaces and Calabi Yau threefolds. A paper by Maria Donten [Don08] provides a complete classification of Kummer 3-folds.

In section 5 we take $d = 2$, i.e. an abelian surface, and we consider the standard representation of the symmetric group S_n in $SL(n-1, \mathbb{Z})$; in this case the Kummer construction will produce the so called generalized Kummer manifolds, $Kum^{(n-1)}$, introduced by Beauville and Fujiki (see [Bea83]) and whose cohomology was already computed by Göttsche and others (see [Göt94]). We explain how to compute cohomology with our methods in this case and we do explicit computation for the

case $n = 4$, i.e. X is a 6-dimensional Kummer manifold (a computation for the case $n = 3$ is done in the last section).

Finally in section 6 we consider a 4-dimensional abelian variety A and we prescribe the action of G only on the tangent space at the unit, i.e we fix the complex representation $\rho_{\mathbb{C}} : G \rightarrow GL(\mathfrak{a})$ and we do not ask a priori that it comes from an integral representation $\rho_{\mathbb{Z}}$. Moreover we consider three special types of groups and representations coming from a recent theorem which characterizes a class of symplectic singularities admitting a (local) symplectic resolution: see section 2.3 where the results in [Bel07] and also in [GK04], [LS08] are summarized. We apply the Kummer construction in this context and in one case we prove it can not lead to a global crepant resolution.

2. NOTATION AND PRELIMINARIES

Throughout the paper we will use the set up introduced above: G is a finite group with representation $\rho_{\mathbb{Z}}$ in $GL(r, \mathbb{Z})$; any extension of ρ will be denoted by a subscript, i.e. $\rho_{\mathbb{C}}$ denotes extension of ρ over \mathbb{C} . Sometimes we will skip the subscript if the context is clear. Next, A is a complex abelian variety of dimension d and G acts on A^r via ρ_A . By $Y = A^r/G$ or A^r/ρ_A we will denote the quotient, which we understand as the space of orbits of the action ρ_A and by $\pi_G : A^r \rightarrow A^r/G$ the quotient morphism.

Moreover $f : X \rightarrow Y = A^r/G$ is assumed to be a crepant resolution of the quotient. That is, X is smooth and projective, f is birational and $K_X = f^*K_Y$.

We denote by \mathbb{Z}_n a cyclic group of order n , by S_n the symmetric group in n letters, or group of permutations of n elements, and by D_{2n} a dihedral group of order $2n$, i.e. semidirect product $D_{2n} = \mathbb{Z}_n \rtimes \mathbb{Z}_2$.

For a group element $g \in G$ by $\langle g \rangle$ we denote the subgroup generated by g . The normalizer of a subgroup $H < G$ is denoted by $N_G(H)$, while $W_G(H)$ stands for the quotient group $N_G(H)/H$. By $[H]_G$ we denote the conjugacy class of H in G , that is the set of subgroups $\{gHg^{-1} < G : g \in G\}$. We skip the subscript whenever the group in which the above objects are defined is clear from the context. The partially ordered set of conjugacy classes of subgroups of G will be denoted by $\mathcal{C}(G)$ with the partial order denoted by \prec . Recall that the cardinality $\#[H]_G$ is equal to the index $[G : N_G(H)]$ and for $H' \in [H]_G$ we have $W_G(H') = W_G(H)$, so this group will be denoted by $W([H]_G)$ and called a Weyl group of H in G . The conjugacy class of an element $h \in G$ is the set $[h]_G = \{ghg^{-1} : g \in G\}$, note that usually $[\langle h \rangle] \neq [h]$.

Let G be a group acting on a set B and $H < G$ a subgroup; by B^H (or $\text{Fix}(H)$) we denote the subset of points of B fixed by H while $B_0^H \subset B^H$ is the set of points whose isotropy (or stabilizer) is exactly H . Clearly, $B^H \setminus B_0^H$ consists of points whose isotropy is bigger than H . We will use repeatedly the fact that if $H' = gHg^{-1}$ then $B^{H'} = gB^H$. In particular, the action of G defines an action of $N(H)$ and of $W(H)$ on B^H which is free on B_0^H .

Let \mathfrak{a} be the tangent space of an abelian variety A at identity, i.e. the Lie algebra of holomorphic vector fields tangent to A , and let $\exp : \mathfrak{a} \rightarrow A$ be the exponential map. The induced action $\rho_{\mathfrak{a}} = \rho_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathfrak{a} : G \rightarrow GL(\mathfrak{a}^r)$ splits into d copies of complexified representation $\rho_{\mathbb{C}} = \rho_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$. The representation $\rho_{\mathfrak{a}}$ is called the tangent action and it is in fact tangent to ρ_A : for $g \in G$ and $p \in A^r$ the derivative of the map $\rho_A(g) : A^r \rightarrow A^r$ at p is $\rho_{\mathfrak{a}}(g) : T_p A^r = \mathfrak{a}^r \rightarrow T_{g(p)} A^r = \mathfrak{a}^r$. Moreover,

if $\exp^r : \mathfrak{a}^r \rightarrow A^r$ is a natural extension of \exp then for every $g \in G$ and $v \in \mathfrak{a}^r$

$$\exp^r(\rho_{\mathfrak{a}}(g)(v)) = \rho_A(g)(\exp^r(v))$$

Let $p \in A^r$ be a point with non-trivial isotropy (or stabilizer) group G_p . In analytic or étale topology the action of G_p around p is equivalent to the induced action $\rho_{\mathfrak{a}|G_p} = d \cdot \rho_{\mathbb{C}|G_p}$ in a neighborhood of 0 in $T_p A^r \simeq \mathbb{C}^{dr}$; this is because $\rho_{\mathfrak{a}|G_p}$ is just the tangent action of G_p at the tangent space $T_p A$.

More generally if we have an endomorphism $g : A \rightarrow A$ with $g(0) = 0$ it has an associated linear analytic representation $\eta_{an}(g) : \mathbb{C}^d \rightarrow \mathbb{C}^d$, which is the corresponding analytic endomorphism on the universal covering or the derivative map at the origin, (see for instance Proposition 1.2.1 in [BL04]). The following is the well-known (holomorphic) Lefschetz fixed point formula for the case of complex tori, as for instance on [BL04].

Theorem 2.1. *Let $g : A \rightarrow A$ be an endomorphism with $g(0) = 0$ and let $\eta_{an}(g) : \mathbb{C}^d \rightarrow \mathbb{C}^d$ as above. The closed analytic subvariety of A consisting of the fixed point of g , denoted by $\text{Fix}(g)$, has dimension equal to the multiplicity of 1 as an eigenvalue of $\eta_{an}(g)$. If it is zero dimensional then $|\text{Fix}(g)| = |\det(1 - \eta_{an}(g))|^2$.*

2.1. Groups and representations. Integral representations of finite groups are fairly well understood; in general we will refer to [New72] and [CR62, Ch. XI] or, for an elementary overview, to [KP02].

For a finite group G we consider the ring $R(G)$ of complex representations of G . By $d \cdot \rho$ we denote the sum of d copies of the representation ρ while by $\rho^{\otimes d}$ and $\rho^{\wedge d}$ we denote d -th tensor and, respectively, alternating power of ρ . Complex representations of rank 1 will be denoted by roots of unity.

We have an additive map $\mu_0 : R(G) \rightarrow \mathbb{Z}$ which to a representation ρ assigns the rank of its maximal trivial subrepresentation. If $\rho_{\mathbb{Z}} \in GL(r, \mathbb{Z})$ is an integral representation of rank r with complexification $\rho_{\mathbb{C}}$ then $r_0 = \mu_0(\rho_{\mathbb{C}})$ is the rank of the maximal trivial subrepresentation of $\rho_{\mathbb{Z}}$ as well, c.f. [CR62, Thm. 73.9]. Indeed, set $\Lambda(\rho_{\mathbb{Z}}) = \{v \in \mathbb{Z}^r : \forall g \in G \ \rho_{\mathbb{Z}}(g)(v) = v\}$ then $\Lambda(\rho_{\mathbb{Z}})$ is a subgroup of \mathbb{Z}^r whose extension to \mathbb{C} is the maximal trivial subrepresentation of $\rho_{\mathbb{C}}$. Moreover, we note that $\Lambda(\rho_{\mathbb{Z}})$ is a saturated, i.e. if $n \cdot v \in \Lambda(\rho_{\mathbb{Z}})$ then $v \in \Lambda(\rho_{\mathbb{Z}})$. Thus the quotient $\mathbb{Z}^r / \Lambda(\rho_{\mathbb{Z}})$ has no torsions, $\rho_{\mathbb{Z}}$ descends to a representation $\eta_{\mathbb{Z}} : G \rightarrow GL(\mathbb{Z}^r / \Lambda(\rho_{\mathbb{Z}})) = GL(r - r_0, \mathbb{Z})$ and we have a G -equivariant exact sequence, with trivial action on the kernel,

$$(2.2) \quad 0 \rightarrow \Lambda(\rho_{\mathbb{Z}}) \simeq \mathbb{Z}^{r_0} \rightarrow \mathbb{Z}^r \rightarrow \mathbb{Z}^r / \Lambda(\rho_{\mathbb{Z}}) \simeq \mathbb{Z}^{r-r_0} \rightarrow 0$$

We will say that $\rho_{\mathbb{Z}}$ is a pullback of $\eta_{\mathbb{Z}}$.

Let S_{r+1} be the symmetric group; a *natural* representation $\nu_{\mathbb{Z}} : S_{r+1} \rightarrow GL(r+1, \mathbb{Z})$ is defined by permuting coordinates. That is, for a point $(e_0, e_1, \dots, e_r) \in \mathbb{Z}^r$ (the choice of coordinates will become clear later) and a permutation $\sigma \in S_{r+1}$ we set

$$\nu_{\mathbb{Z}}(\sigma)(e_0, \dots, e_r) = (e_{\sigma(0)}, \dots, e_{\sigma(r)})$$

In other words, $\nu_{\mathbb{Z}}(S_{r+1})$ is generated by elementary matrices E_{ij} responsible for transposing i -th and j -th vectors of the chosen basis. Clearly, $(E_{ij}^T)^{-1} = E_{ij}$, where T denotes transposition of the matrix, so this representation is isomorphic to its dual. The natural representation of S_{r+1} contains a fixed subspace $e_0 = \dots = e_r$ and thus, as above in 2.2, $\nu_{\mathbb{Z}}$ is a pull-back of a *quotient* representation $\eta_{\mathbb{Z}} : S_{r+1} \rightarrow GL(r, \mathbb{Z})$. On the other hand, we have a S_{r+1} -invariant

subspace $e_0 + \cdots + e_r = 0$ which defines a representation $\rho_{\mathbb{Z}} : S_{r+1} \rightarrow GL(r, \mathbb{Z})$ which we call *standard* representation. Hence the standard representation, being the kernel of $(e_0, \dots, e_r) \mapsto e_0 + \cdots + e_r$, is dual (as \mathbb{Z} -module) to quotient representation. In addition, the S_{r+1} -equivariant composition of inclusion and quotient $\mathbb{Z}^r \hookrightarrow \mathbb{Z}^{r+1} \rightarrow \mathbb{Z}^r$ makes $\rho_{\mathbb{Z}}$ a subrepresentation of $\eta_{\mathbb{Z}}$ with torsion cokernel and induces an isomorphism for complexifications $\rho_{\mathbb{C}} = \eta_{\mathbb{C}}$ which implies splitting of complex representation $\nu_{\mathbb{C}} = \mathbf{1}_{\mathbb{C}} + \rho_{\mathbb{C}}$. However, $\eta_{\mathbb{Z}}$ is not conjugate to $\rho_{\mathbb{Z}}$ in $GL(r, \mathbb{Z})$, c.f. [CR62, p. 505].

Let $G_{r,m} := \mathbb{Z}_m^r \rtimes S_r$ be the semidirect product, where S_r acts on \mathbb{Z}_m^r by permutations. We have a natural action of G_r on \mathbb{C}^r , namely the group \mathbb{Z}_m^r acts on \mathbb{C}^r diagonally and S_r by permutations of the coordinates. We have a sequence of quotients

$$(2.3) \quad \mathbb{C}^r \longrightarrow (\mathbb{C}/\mathbb{Z}_m)^r \longrightarrow (\mathbb{C}/\mathbb{Z}_m)^r / S_r = \mathbb{C}^r / G_{r,m}.$$

Let us finally describe a special group, the binary tetrahedral group T , and its (complex) representations; we follow the description given in [LS08]. T is the preimage of the symmetric group of a regular tetrahedron, T_0 , via the natural map $SU(2) \rightarrow SO(3)$. As a subgroup of $SU(2)$ it is generated by the elements

$$I = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad \text{and} \quad \tau = -\frac{1}{2} \begin{pmatrix} 1+i & -1+i \\ 1+i & 1-i \end{pmatrix}.$$

It can also be described as the semidirect product $Q_8 \rtimes \mathbb{Z}_3$ of the quaternion group $Q_8 = \{\pm 1, \pm I, \pm J, \pm K\}$ and the cyclic group of order 3. This group has 7 irreducible complex representations, we are interested in the two dimensional ones. The standard arising from the embedding $T \subset SU(2)$, ρ_0 , and other two being $\rho_j := \rho_0 \otimes \mathbb{C}_j$, for $j = 1, 2$, where \mathbb{C}_j is the 1-dimensional representation given by the multiplication by a third root of unity ϵ_3^j . The last two are dual to each other.

2.2. Crepant resolutions. Working with Kummer construction requires the use of crepant resolutions of quotient Gorenstein singularities. From dimension 4 on they are far from being understood. The following is a list of references which we found useful, it is of course not exhaustive: [Rei87], [DHz06], [Rei02] for crepant resolution of canonical singularities and [GK04], [Fu06] for the symplectic case.

In what follows we will use known facts about resolving 2 dimensional singularities, for which we refer to [BPVdV84]. We will also use the following elementary observation about constructing consecutive resolutions.

Lemma 2.4. *Consider finite subgroups $H < G < SL(n, \mathbb{C})$, where $H \triangleleft G$ is a normal subgroups of G . Let $X_H \rightarrow \mathbb{C}^n/H$ be a resolution of the quotient singularities which is G/H -equivariant. That is, the quotient group G/H acts on \mathbb{C}^n/H and assume that this action lifts up to X_H . If $X_G \rightarrow X_H/(G/H)$ is a resolution then the composition with the induced map $X_G \rightarrow X_H/(G/H) \rightarrow \mathbb{C}^n/G$ is a resolution of the quotient singularity. If both intermediate resolutions are crepant then the resulting resolution is crepant as well.*

Finally one of our tools will be McKay correspondence which allows to describe the structure of a crepant resolution of a quotient singularity in terms of the conjugacy classes, or representations, of the groups itself, see [Rei02], for an exposition in this regard, as well as [Kal02], [GK04].

2.3. Symplectic case. Let G be a finite group with a complex representation $\rho_{\mathbb{C}} : G \rightarrow GL(V)$. Then G has a symplectic representation $\rho \oplus \rho^* : G \rightarrow Sp(V \oplus V^*)$, where ρ^* is the dual representation: the symplectic form preserved is given by the identity in $V \otimes V^*$. Moreover if the representation $\rho_{\mathbb{C}}$ preserves a non degenerate symmetric 2-form on V then there is a G -equivariant isomorphism $V \simeq V^*$; in this case therefore $2\rho_{\mathbb{C}} : G \rightarrow Sp(V \oplus V)$ is a symplectic representation. A primary example is the case when G is a Weyl group acting on the lattice of roots of a simple Lie algebra: the action of G preserves the Killing form (see [Bou68]).

Let $\rho : G \rightarrow Sp(V)$ be a complex symplectic representation. The symplectic form σ on V descends to a symplectic form $\hat{\sigma}$ on the regular part of V/G . A proper morphism $f : X \rightarrow V/G$ is a symplectic resolution if X is smooth and $f^*(\hat{\sigma})$ extends to a symplectic form on X .

More generally a normal variety Y is called a symplectic variety if its smooth part admits a holomorphic symplectic form σ whose pull back to any resolution $f : X \rightarrow Y$ extends to a holomorphic 2-form on X . If this extended holomorphic two form is a symplectic form then f is called a symplectic resolution. Note that if Y is a symplectic variety and $f : X \rightarrow Y$ is a resolution, then f is symplectic if and only if it is crepant (see for instance proposition 1.3 in [Fu06]).

Symplectic resolutions are very rare and they have been considered by a number of people; the following two results give necessary conditions for the existence of a symplectic resolution for quotient singularities. They were proved in [Ver00] and [Kal03]; see the sections 3 and 4 of the survey [Fu06] also for appropriate references.

Proposition 2.5. *Let X be a smooth irreducible symplectic variety and G a finite group of symplectic automorphisms on X . Assume that $Y = X/G$ admits a symplectic resolution, then the subvariety $F = \bigcup_{g \neq 1} \text{Fix}(g) \subset X$ is either empty or of pure codimension 2 in X . In particular if Y has an isolated symplectic singularity, then it admits a symplectic resolution only if it is of dimension 2.*

Proposition 2.6. *Let $V = \mathbb{C}^{2n}$ and G a finite group of symplectic automorphisms on V , i.e. $\rho_{\mathbb{C}} : G \rightarrow Sp(\mathbb{C}^{2n})$. Assume that $Y = V/G$ admits a symplectic resolution, then G is generated by symplectic reflections, i.e. elements g such that $\text{codim}(\text{Fix}(g)) = 2$. In the special case in which $\rho_{\mathbb{C}} = \eta_{\mathbb{C}} \oplus \eta_{\mathbb{C}}^* : G \rightarrow Sp(\mathbb{C}^n \oplus \mathbb{C}^{n*})$ is a sum of a complex representation $\eta_{\mathbb{C}}$ and its dual, then $V \oplus V^*/G$ has a symplectic resolution if $\eta_{\mathbb{C}} : G \rightarrow GL(\mathbb{C}^n)$ is generated by complex reflections (i.e. elements g such that $\text{codim}(\text{Fix}(g)) = 1$).*

Recently the following necessary and sufficient condition has been proved in [GK04] and [Bel07]; see also [LS08].

Proposition 2.7. *Let G be a finite group, $\rho_{\mathbb{C}} : G \rightarrow GL(\mathbb{C}^n)$ an irreducible complex representation and assume that $\rho_{\mathbb{C}}(G)$ is generated by complex reflections. Then $V \oplus V^*/G$ has a symplectic resolution if and only if $(G, \rho_{\mathbb{C}})$ is one of the following:*

- 1) S_{n+1} and $\rho_{\mathbb{C}}$ is the standard representation.
- 2) $G_{n,m} = \mathbb{Z}_m^n \rtimes S_n$ and $\rho_{\mathbb{C}}$ is the natural representation described in the previous section.
- 3) $Q_8 \rtimes \mathbb{Z}_3$, the binary tetrahedral group T , $n = 2$ and $\rho_{\mathbb{C}}$ is the representation ρ_1 described in the previous section.

The first case corresponds to the Weyl groups of type A , namely the Weyl groups of the Lie algebra \mathfrak{a}_n . The second case, for $m = 2$, corresponds to the Weil groups

of type B and C (see [Bou68], Chapter VI, Tables I, II and III). Note that in the first case and in the second when $m = 2$ the representation is integral.

A local symplectic resolution is obtained in 1) and 2) via Hilbert schemes: namely for a smooth surface S the Hilbert scheme $Hilb^n(S)$ provides a crepant resolution $Hilb^n(S) \rightarrow Sym^n(S)$. In 1) consider the null-fiber of the morphism which is the composition

$$Hilb^{n+1}(\mathbb{C}^2) \rightarrow Sym^{n+1}(\mathbb{C}^2) \rightarrow \mathbb{C}^2$$

where $s : Sym^{n+1}(\mathbb{C}^2) \rightarrow \mathbb{C}^2$, is the the summation $\sum_{i=0, \dots, n} e_i^j$ for $j = 1, 2$; note that $s^{-1}(0) = \mathbb{C}^{2n}/S_n$.

In 2) consider first a minimal resolution of the A_{m-1} singularity $\mathbb{C}^2/\mathbb{Z}_m$, namely $\widehat{\mathbb{C}^2/\mathbb{Z}_m} \rightarrow \mathbb{C}^2/\mathbb{Z}_m$, and then the composition

$$Hilb^n(\widehat{\mathbb{C}^2/\mathbb{Z}_m}) \rightarrow Sym^n(\widehat{\mathbb{C}^2/\mathbb{Z}_m}) \rightarrow Sym^n(\mathbb{C}^2/\mathbb{Z}_m).$$

An explicit local resolution for the third case has been given recently in [LS08].

The Kummer construction applied to an abelian surface for the group and the representation in 1) of Proposition 2.7 gives a generalized Kummer variety $Kum^n(A)$, as constructed by Beauville, see [Bea83] and also [Fuj83]. A global resolution $f : X \rightarrow Y = A^n/S_{n+1}$ is obtained, as in the local case, considering the null-fiber of the composition $Hilb^{n+1}(A) \rightarrow A^{n+1}/S_{n+1} \rightarrow A$. The first map is the Hilbert-to-Chow map where the action of S_{n+1} on A^{n+1} is by permutation of the coordinates (e_0, \dots, e_n) , the quotient is interpreted as the Chow variety of A ; the second is the summation $\sum_{i=0, \dots, n} e_i$.

In the case 2) of Proposition 2.7, with $m = 2$ (the case of integral representation), the Kummer construction applied to an Abelian surface gives the other series of symplectic manifolds considered by Beauville, see [Bea83] and also [Fuj83].

Namely the group \mathbb{Z}_2^n acts on A^n diagonally so we have a sequence of quotients

$$(2.8) \quad A^n \longrightarrow (A/\mathbb{Z}_2)^n \longrightarrow (A/\mathbb{Z}_2)^n/S_n$$

and since the first quotient can be desingularized as $(Kum^1(A))^n$ then the latter quotient has a natural desingularization as $Hilb^n(Kum^1(A))$, as follows from lemma 2.4.

In the last section we will prove that a (generalized) Kummer construction in the case 3) of Proposition 2.7 cannot have a global crepant resolution.

3. COMPUTING COHOMOLOGY

In this paper we will calculate the De Rham cohomology, or the Betti numbers $b_i(X) = \dim_{\mathbb{C}} H^i(X, \mathbb{C})$, for some varieties which are the results of Kummer constructions in the low dimensional cases.

This is a two step process which consists of: (1) calculating cohomology of $Y = A^r/G$ and (2) calculating the contribution coming from resolution $X \rightarrow Y$.

Recall that one can assign to any complex algebraic variety X , not necessarily smooth, or compact, or irreducible, a virtual Poincaré polynomial, $P_X(t)$, with the following properties. For a compact manifold X of complex dimension n the polynomial is defined as the standard Poincaré polynomial

$$P_X(t) = \sum_{i=0}^{2n} b_i(X) t^i \in \mathbb{Z}[t],$$

where t is a formal variable and $b_i(X) = \dim H_{DR}^i(X)$ are the Betti numbers. Moreover if Y is a closed algebraic subset of X and $U := X \setminus Y$ then

$$P_X(t) = P_Y(t) + P_U(t).$$

For further details we refer to [Ful93, 4.5] and [Tot02, 2]. We remark that the virtual Poincaré is actually the standard Poincaré polynomial also if X is compact and has quotient singularities, see [Ful93, p. 94].

3.1. Quotients. Computing cohomology of the quotient is pretty straightforward: we look at $H^i(A^r, \mathbb{C})$ or $H^{pq}(A^r)$ on which G acts via representation $(2d \cdot \rho_{\mathbb{C}})^{\wedge i}$ or $(d \cdot \rho_{\mathbb{C}})^{\wedge p} \otimes (d \cdot \rho_{\mathbb{C}})^{\wedge q}$, respectively. Note that $\bar{\rho}_{\mathbb{C}} = \rho_{\mathbb{C}}$, because ρ is real. By looking at the identity component of this representation one determines the dimension of the space of G -invariant forms which subsequently can be used to get the information about the cohomology of $Y = A^r/G$, see e.g. [Bre72, Ch. III]. We note that products of representations of finite groups can be calculated in a standard way by looking at their characters, see e.g. [FH91, Part I].

Given an action of a group G on a variety Z , in order to formulate the result in terms of Poincaré polynomial, we define a G -Poincaré polynomial, $P_{Z,G}(t) \in R(G)[t]$, whose coefficient at t^i is equal to the representation of the induced G -action on the vector space $H^i(Z, \mathbb{C})$. In particular, in our set-up

$$(3.1) \quad P_{A^r,G}(t) = \sum_{i=0}^{2rd} (2d \cdot \rho_{\mathbb{C}})^{\wedge i} \cdot t^i;$$

we will denote this polynomial by $(1+t)^{2d\rho}$.

Lemma 3.2. *For $Y = A^r/\rho_A$ we have $P_Y(t) = \mu_0((1+t)^{2d\rho})$ where $\mu_0 : R(G)[t] \rightarrow \mathbb{Z}[t]$ is the reduction of coefficients via $\mu_0 : R(G) \rightarrow \mathbb{Z}$.*

3.2. Resolution. For understanding the cohomology of the resolution $X \rightarrow Y$ we will write the quotient $Y = A^r/G$ as a disjoint sum of locally closed sets (strata) $Y([H])$ consisting of orbits of points whose isotropy is in the conjugacy class of a subgroup $H < G$. The calculation is somehow in the spirit of [HH90]. Over $Y([H])$ the singularities of Y will be locally quotients of \mathbb{C}^{rd} by action of H . Thus, by taking inverse images of sets $Y([H])$ we will produce a decomposition of X into a disjoint sum of locally closed sets $X([H])$ such that the restriction $X([H]) \rightarrow Y([H])$ will be a locally trivial fiber bundle with a fiber $F([H])$ depending on the resolution of the H -quotient singularity. Now the cohomology of X will be computed by looking at each of $X([H])$ and using virtual Poincaré polynomial for each of them.

The set $Y([H])$ may be disconnected, as for instance in the case of a Kummer surface construction. However, as already noted, for every $y \in Y([H])$ the singularity of Y in a neighborhood of y is of type $\mathbb{C}^{rd}/d\rho_{\mathbb{C}}(H)$. That is, given $p \in A^r$ in the orbit represented by $y \in Y([H])$ with isotropy $G_p = H$, there is a map $T_p A^r/\rho_{\mathfrak{a}}(H) = \mathfrak{a}^r/\rho_{\mathfrak{a}}(H) \rightarrow Y$ which is an isomorphism of analytic neighborhoods of the 0 orbit and of y . Indeed, consider evaluation of vector fields at p , that is $\exp_p : \mathfrak{a}^r = T_p A^r \rightarrow A^r$, where $\exp_p(0) = p$; it is H -equivariant and thus it defines a map of quotients $T_p A^r/\rho_{\mathfrak{a}}(H) \rightarrow A^r/\rho_A(H)$. Compose it with the natural map of spaces of orbits $\pi_{G/H} : A^r/H \rightarrow A^r/G$, coming from the inclusion $H < G$. The first of these morphism is an isomorphism of analytic neighborhoods of 0 and p . On the other hand, we can choose an analytic open neighborhood U of p such that $gU = U$ for $g \in H$ and $gU \cap U = \emptyset$ for $g \notin H$. Thus the orbits of G and H restricted to U

coincide, that is $\pi_{G|U} = \pi_{H|U}$, for the respective orbit class maps. Hence $\pi_{G/H}$ is bijective and, by normality of quotients, an isomorphism over the respective neighborhoods of $\pi_H(p)$ and y .

By these arguments, there exists an open analytic neighborhood V of $\pi_H((A^r)_0^H)$ such $\pi_{G/H}$ restricted to V is a local isomorphism onto an open neighborhood of $Y([H])$.

Now let $r_0 = \mu_0(\rho|_H)$ be the rank of the maximal trivial subrepresentation of $(\rho_{\mathbb{C}})|_H$ so that, as in formula 2.2, $(\rho_{\mathbb{Z}})|_H$ is a pull-back of $\eta_H : H \rightarrow SL(r_H, \mathbb{Z})$ of rank r_H , with $r_H + r_0 = r$, and η has no non-trivial fixed point. Accordingly, after extending to A , we have a H -equivariant sequence of abelian varieties

$$(3.3) \quad A^{r_0} \hookrightarrow A^r \longrightarrow A^{r_H}$$

where the action of H on A^{r_H} has only a finite number of fixed points. Thus $(A^r)^H$ is a union of (affine) abelian subvarieties of A^r of dimension dr_0 .

We fix a resolution of the quotient singularity $\mathbb{C}^{dr_H}/d\eta_{\mathbb{C}}$ with the special (central) fiber $F(H)$. Then it determines a product resolution of a neighborhood of a component of $(A^r)^H$ in A^r/H . Indeed, as follows from the preceding discussion, any such component has a neighborhood isomorphic to a neighborhood of $A^{r_0} \times \{0\}$ in $A^{r_0} \times \mathbb{C}^{dr_H}/d\eta_{\mathbb{C}}$, hence it admits a product resolution. We will consider resolutions $X \rightarrow Y$ which are locally product in the following sense.

Definition 3.4. Let $f : X \rightarrow Y = A^r/G$ be a resolution of singularities. We say that it is a *locally product* if for every $H < G$ and every irreducible component K of $Y([H])$ there exists an open analytic neighborhood $U \subset Y$ of K such that the pull-back via $\pi_{G/H}$ of the resolution $f^{-1}(U) \rightarrow U$ over an open subset of A^r/H is analytically equivalent to a product resolution with the special fiber $F([H])$ depending only on the conjugacy class of H .

The next result provides a description of both $Y([H])$ and $X([H])$ in terms of $(A^r)_0^H$ and the group $W(H)$.

Lemma 3.5. *Let $f : X \rightarrow Y = A^r/G$ be a locally product resolution of singularities. Then $f|_{X([H])} : X([H]) \rightarrow Y([H])$ is an étale fiber bundle whose fiber $F([H])$ is isomorphic to the special fiber of a resolution of the quotient singularity \mathbb{C}^{dr_H}/H . Moreover, let $\overline{Y}([H]) \subset Y$ denote the closure of $Y([H])$ in Y and $\widehat{Y}([H]) \rightarrow \overline{Y}([H])$ be its normalization. Then the following holds*

- The action of $N(H)$ determines an action of $W(H)$ on $\overline{(A^r)_0^H}$ and the morphism $\overline{(A^r)_0^H} \rightarrow \widehat{Y}([H])$ is the quotient by $W(H)$.
- The action of $W([H])$ on $\overline{(A^r)_0^H}$ lifts to the product $\overline{(A^r)_0^H} \times F([H])$ in such a way that there is a commutative diagram

$$\begin{array}{ccccc} \overline{(A^r)_0^H} \times F([H]) & \longrightarrow & \left(\overline{(A^r)_0^H} \times F([H]) \right) / W([H]) & \longleftarrow & X([H]) \\ \downarrow & & \downarrow & & \downarrow \\ \overline{(A^r)_0^H} & \longrightarrow & \widehat{Y}([H]) & \longleftarrow & Y([H]) \end{array}$$

where the horizontal arrows on the left hand side are quotient maps while these on the right hand side are inclusions onto open subsets

Proof. Take the quotient map $A^r \rightarrow Y = A^r/G \supset Y([H])$ and consider inverse image of $Y([H])$. The inverse image decomposes into disjoint sets $(A^r)_0^{H'}$, depending on the isotropy class $H' \in [H]$. The normalizer $N(H)$ acts on the set of points whose isotropy is H , i.e. $(A^r)_0^H$, and this determines a free action of $W(H)$ on this set of points. Take factorization of the quotient map π_G into $A^r \rightarrow A^r/W(H) \rightarrow A^r/G$ which gives a regular birational map $(A^r)_0^H/W(H) \rightarrow \overline{Y([H])}$. This proves the central statements of the lemma. The rest follows because of our assumption regarding the resolution. \square

We note that the map $(A^r)_0^H \times F([H]) \rightarrow X([H])$ usually does not extend, so that $\overline{(A^r)_0^H} \times F([H])/W([H])$ is not the normalization of the closure of $X([H])$.

We will use the preceding lemma to calculate the Poincaré polynomial of both $Y([H])$ and $X([H])$. In fact, the following is how one computes the cohomology of $\overline{Y([H])}$ and $\overline{(A^r)_0^H} \times F([H])/W([H])$. Let $K \subset Y([H])$ be an irreducible component whose normalized closure we denote by \widehat{K} . Then $\widehat{K} \simeq A_K/W_K$ where $W_K < W(H)$ is the subgroup which preserves $A_K \simeq A^{r_0}$, a component of the closure of $(A^r)_0^H$ which dominates K . Thus, as in 3.1, we can write

$$(3.6) \quad P_{A_K, W_K}(t) = \sum_{i=0}^{2dr_0} (2d \cdot \eta_K)^{\wedge i} \cdot t^i = (1+t)^{2d\eta_K}$$

where $\eta_K : W_K \rightarrow GL(r_K, \mathbb{C})$ is a representation of W_K induced from $\rho_{\mathbb{C}}$. That is, the group $N(H)$, and thus $W(H)$, acts on the fixed point space of $(\rho_{\mathbb{C}})|_H$, as in 2.2, hence it yields an action of $W_K < W(H)$. Therefore, as in 3.2, we get $P_{\widehat{K}} = \mu_0(P_{A_K, W_K})$.

Now, recall that the McKay correspondence postulates a canonical relation of conjugacy classes of elements in a group H with cohomology or homology of a crepant resolution of its quotient singularity, see e.g. [Rei02], [BM94], [Kal02] and [GK04]. Thus, if the McKay correspondence holds for the fixed resolution of $\mathbb{C}^{r_H d}/H$ then we can use it to understand the action of $W(H)$ on cohomology of $\overline{(A^r)_0^H} \times F(H)$. Indeed, the group $W(H)$ acts on the cohomology of $F(H)$ as $W(H)$ acts on the conjugacy classes of H . So, in the situation introduced in the previous paragraph, the W_K -Poincaré polynomial $P_{F(H), W_K}$ is determined by the adjoint action of W_K on conjugacy classes of elements in H , which is $w([h]_H) \mapsto [whw^{-1}]_H$, where $w \in N(H)$ represents an element of $W(H)$ and $h \in H$. Thus, whenever the representation ρ is fixed, we will simply write P_{H, W_K} instead of $P_{F(H), W_K}$. We conclude

$$(3.7) \quad P_{(A_K \times F(H)), W_K} = (1+t)^{2d\eta_K} \cdot P_{H, W_K}$$

However, deriving from it the virtual Poincaré polynomial for $X([H])$ requires understanding lower dimensional strata, that is the quotient $(A_K \times F(H))/W_K$ over the difference $\overline{Y([H])} \setminus Y([H])$. To this end, let $H' > H$ be a subgroup and $K' \subset \overline{K}$ an irreducible component of $Y([H'])$. By $W_{K; K'} < W_K$ we denote the subgroup of W_K which preserves K' . Then, by restricting the representations we get $P_{H, W_{K; K'}}$, which is the $W_{K; K'}$ -polynomial describing the action of $W_{K; K'}$ of the cohomology of the fiber $F(H)$. On the other hand we have an induced representation $\eta_{K; K'} : W_{K; K'} \rightarrow GL(r_{K'}, \mathbb{C})$ and therefore the action of $W_{K; K'}$

on the cohomology of the torus $A_{K'} \simeq A^{r_{K'}}$ dominating K' is described by the polynomial $(1+t)^{2d\eta_{K;K'}}$, thus

$$(3.8) \quad P_{(A_{K'} \times F(H)), W_{K;K'}} = (1+t)^{2d\eta_{K;K'}} \cdot P_{H, W_{K;K'}}$$

The following statement is a summary of the preceding discussion.

Principle 3.9. *Let X be obtained via a Kummer construction as described in the introduction; i.e. X is a crepant resolution of A^r/G satisfying a) and b) in the introduction. The Poincaré polynomial $P_X \in \mathbb{Z}[t]$ is described by the following formula*

$$\sum_{[H] \in \mathcal{C}(G)} \sum_{K \subset Y([H])} \left[\mu_0((1+t)^{2d\eta_K} \cdot P_{H, W_K}) - \sum_{[H'] \succ [H]} \sum_{K' \subset \overline{K} \cap Y([H'])} a_{K;K'} \cdot \mu_0((1+t)^{2d\eta_{K;K'}} \cdot P_{H, W_{K;K'}}) \right]$$

where K runs through all irreducible components of $Y([H])$ and K' through all irreducible components of $\overline{K} \cap Y([H'])$ and $a_{K;K'}$ are numbers which depend on the incidence of the closures of components of the stratification of Y .

The data which appears in the above formula is of two types: (1) depending on the group G and its complex representation $\rho_{\mathbb{C}}$ and (2) depending on the integral conjugacy class of the representation $\rho_{\mathbb{Z}}$ which determines the geometry of the quotient Y and its stratification. As it is shown in [Don08], already in dimension three Kummer constructions with the same $\rho_{\mathbb{C}}$ can have different $\rho_{\mathbb{Z}}$ and different cohomology.

4. BUILDING UPON ELLIPTIC CURVES: $d = 1$

In the present section A denotes an elliptic curve. Quotients of products of elliptic curves by actions of specific groups have been considered by several people: [PR05], [CH07], [CS07].

4.1. Special Kummer surfaces: $r = 2$. Let us start with the following easy classical case. The classification of rank 2 groups whose action give Gorenstein singularities, known as Du Val singularities, is very well understood, these are finite subgroups of $SL(2, \mathbb{C})$, [Dur79]. On the other hand there are only 4 types of nontrivial subgroups of $SL(2, \mathbb{Z})$, all are cyclic and generated, up to conjugation in $GL(2, \mathbb{Z})$, by one of the following matrices (cf. [New72, Ch. IX]).

$$(4.1) \quad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$$

They generate cyclic groups \mathbb{Z}_2 , \mathbb{Z}_3 , \mathbb{Z}_4 and \mathbb{Z}_6 .

Let us discuss the case of $\rho_{\mathbb{Z}} : \mathbb{Z}_6 \rightarrow SL(2, \mathbb{Z})$. In the following table we list its subgroups, each of them generated by an element g . For each of them we give the number of its fixed points and the number of singular points whose isotropy is exactly the group in question. The latter number is obtained by subtracting those fixed points whose isotropy is bigger and dividing by the cardinality of the respective orbit of G , which is the index of the subgroup in question. In the last two columns we present the Dynkin diagram of the special fiber of the minimal resolution of the respective singular point and its virtual Poincaré polynomial.

g	# fix pts	# sing pts	resolution	Poincaré
$\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$	1	1	$\bullet\text{---}\bullet\text{---}\bullet\text{---}\bullet\text{---}\bullet$	$1 + 5t$
$\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$	9	4	$\bullet\text{---}\bullet$	$1 + 2t$
$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	16	5	\bullet	$1 + t$

On the other hand we note that over the complex number, that is in $SL(2, \mathbb{C})$, the representation ρ (or, equivalently, the matrix generating the image of ρ) is equal to diagonal representation $\epsilon_6 + \epsilon_6^5$, where ϵ_6 denotes sixth primitive root of unity. Thus we compute $\rho \otimes \rho = 2 \cdot \mathbf{1} + \epsilon_6^2 + \epsilon_6^4$ hence the space of invariant $(1, 1)$ forms is of dimension 2. We add to it the contribution of cohomology coming from resolving singular points of the quotient, as listed above, to get

$$2 + 1 \times 5 + 4 \times 2 + 5 \times 1 = 20$$

which is the dimension of H^{11} for a K3 surface.

4.2. Kummer threefolds: $r = 3$. Construction of Calabi-Yau threefolds via quotients have been considered in e.g. [CH07], [CS07] as well as in [OS01]. We note, however, that the last reference concerns dividing abelian varieties by an action of translations so that the quotient map is étale. As for the linear action, which is used in the present paper, there is a classical book [New72, Ch. IX] which provides a list of isomorphism classes of finite subgroups of $SL(3, \mathbb{Z})$. Donten, [Don08] classifies noncyclic finite subgroups of $SL(3, \mathbb{Z})$ up to conjugacy in $GL(3, \mathbb{Z})$. The following proposition summarizes these results.

Proposition 4.2. *The following are, up to isomorphism, (non-trivial) finite subgroups of $SL(3, \mathbb{Z})$:*

- cyclic groups \mathbb{Z}_a , of rank a , for $a = 2, 3, 4$ and 6 ,
- dihedral groups D_{2a} , of rank $2a$, for $a = 2, 3, 4$ and 6 , which have, respectively, 4, 3, 2 and 1 conjugacy classes in $GL(3, \mathbb{Z})$
- the alternating group A_4 which has 3 conjugacy classes in $GL(3, \mathbb{Z})$ (e.g. the tetrahedral group of isometries of the tetrahedron),
- the symmetric group S_4 which has 3 conjugacy classes in $GL(3, \mathbb{Z})$ (e.g. octahedral group of isometries of a cube)

Lemma 4.3. *For a non-identity matrix $M \in SL(3, \mathbb{Z})$ of finite order the fixed point set of M is of dimension one.*

Proof. The eigenvalues of M are roots of unity and their product is 1 and at least one of them is a real number, hence equal ± 1 . If λ_1 and λ_2 are non-real eigenvalues then, as roots of a degree 3 real polynomial, they are conjugate hence their product is 1. Thus, one of the eigenvalues is 1 and the eigenspace of 1 is either of dimension 1 or 3. \square

In particular the case of cyclic groups is not allowed since we assume that the fixed point set of G is $\{0\}$.

Consider the case of $D_4 = \mathbb{Z}_2 \times \mathbb{Z}_2$, generated by matrices

$$A_{001} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad A_{010} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad A_{100} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

In toric terms the quotient map $\mathbb{C}^3 \rightarrow \mathbb{C}^3/D_4$ is given by extending the standard integral lattice by adding generators $(1/2, 1/2, 0)$, $(1/2, 0, 1/2)$ and, subsequently, also $(0, 1/2, 1/2)$. It admits two types of resolution. The first one is obtained by consecutive blow-up of singularities of \mathbb{Z}_2 actions, as in 2.4, while the second one is invariant with respect to permutation of coordinates. They differ by a flop. The picture below presents fans of both of them as of a section of the first octant. Namely, we present the respective divisions of the standard cone (first octant) with vertexes (standard basis) denoted by \circ and the boundary presented by dotted lines. Next, \bullet denote exceptional divisors of the resolution and solid line segments stand for the 2 dimensional cones of the resolution.



We note that the right-hand side resolution is invariant with respect to the action of permutations of coordinates of the standard cone thus this resolution satisfies assumptions of lemma 2.4 with respect to $D_4 \triangleleft A_3 = D_4 \rtimes \mathbb{Z}_3$ or $D_4 \triangleleft S_4 = D_4 \rtimes S_3$. Let us note that the resolution of singularities is uniquely defined in codimension 2, hence locally product in the sense of definition 3.4.

Finally note also that in this 3-fold case it is not hard to construct a global crepant resolution. Namely one first constructs a resolution of the 1-dimensional singular strata: take an element in the isotropy group of a point in this strata, g , and consider $Fix(g) = \bigcup C_i$. Let C_0 be the component through the origin of A^3 ; this is a one dimensional abelian variety which is also a subgroup of A^3 . Take the quotient $\pi : A^3 \rightarrow A^3/C_0$ and take the induced action of g on the surface A^3/C_0 : resolve the singularities of the quotient $(A^3/C_0)/g$ which are the image of the curves C_i under π . The lift up of this surface desingularization via π will give a desingularization of A^3/G along the strata C_i . After that one glues the desingularization of the isolated singularities.

4.3. Cohomology, case of the octahedral group. In this section we discuss, as an example, the case of the octahedral group which is just a representation of S_4 into $SL(3, \mathbb{Z})$. We note that S_4 admits other representations in $SL(3, \mathbb{Z})$ which are not conjugate in $GL(3, \mathbb{Z})$ to the one which will be considered (the other two can be found in [Don08]).

The following table summarizes information about singularities of A^3/G in codimension 2. In the first column, we write down conjugacy classes of non-trivial elements g of this group together with equations of their fixed points in A^3 with coordinates (e_1, e_2, e_3) (column 2). In each case, because of lemma 4.3, the fixed point set is a number of elliptic curves, so in the next column we write the number of components of the fixed point set. Next, we write the group generated by this element $\langle g \rangle$ together with $W(g) := N(\langle g \rangle)/\langle g \rangle$. The group $W(g)$ acts on the fixed point set of g and only in the first case it acts nontrivially on the set of components while its action on each component (elliptic curve) is an involution $e \mapsto -e$ hence it has 4 fixed points. Thus, by dividing by action of $W(g)$, we get the set of singular points of the quotient whose generic points (in each component) have isotropy $\langle g \rangle$, we write it in the next column. In the last column we write the virtual $W(g)$ -Poincaré polynomial of the fiber of a minimal resolution of the respective

singularity. The polynomial provides the information about the action of $W(g)$, that is ϵ is the representation satisfying $\epsilon^2 = 1$. We note that since the minimal resolution in case of surfaces is unique any such resolution will be locally product in the sense of definition 3.4 hence lemma 3.5 can be applied.

g	$Fix(g)$	# cmpnts	$\langle g \rangle$	$W(g)$	$\widehat{Y(\langle g \rangle)}$	Poincaré
$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$2e_1 = 0$ $2e_2 = 0$	16	\mathbb{Z}_2	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$6 \times \mathbb{P}^1$	$1 + t^2$
$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$e_1 = e_2$ $2e_1 = 0$	4	\mathbb{Z}_4	\mathbb{Z}_2	$4 \times \mathbb{P}^1$	$1 + (2 + \epsilon)t^2$
$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	$e_1 = e_2$ $2e_3 = 0$	4	\mathbb{Z}_2	\mathbb{Z}_2	$4 \times \mathbb{P}^1$	$1 + t^2$
$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$	$e_1 = e_2$ $e_1 = e_3$	1	\mathbb{Z}_3	\mathbb{Z}_2	$1 \times \mathbb{P}^1$	$1 + (1 + \epsilon)t^2$

Note that the components of the fixed point sets listed in the above table meet in a set $\{p \in A^3 : 2p = 0\} = \{(e_1, e_2, e_3) \in A^3 : 2e_1 = 2e_2 = 2e_3 = 0\}$ of cardinality $4^3 = 64$ which is where are located points with non-cyclic isotropy groups. We list them in the subsequent table, together with the virtual Poincaré polynomial of the fiber of a crepant resolution, which by McKay correspondence is related to the number of conjugacy classes of the respective group.

subgroup	fixed set in $\{2p = 0\}$	# fixed pts	# sing pts	Poincaré
D_4	$e_1 \neq e_2 \neq e_3 \neq e_1$	24	4	$1 + 3t^2$
$3 \times D_8$	$e_i = e_j \neq e_k, \{i, j, k\} = \{1, 2, 3\}$	36	12	$1 + 4t^2$
$G = S_4$	$e_1 = e_2 = e_3$	4	4	$1 + 4t^2$

Now we pass to computing the Poincaré polynomials of the respective strata $Y([H])$. We write our calculation in typewriter type, in the form of code of `maxima`, [Sch07]. We start with the generic strata, that is $Y([id])$. This is obtained by subtracting from the Poincaré polynomial of the quotient A^3/S_4 which we calculate by looking at the invariants of the respective representation, as in [FH91, Part I], the singular locus. The latter consists of 16 copies of \mathbb{P}^1 , each with 4 points removed, and 16 points associated to non-cyclic subgroups. The result is the polynomial of 3 dimensional stratum

`S3(t):=1+t^2+4*t^3+t^4+t^6-(15*(1+t^2-4)+20);`

Next we consider 1-dimensional strata which are associated to cyclic groups of type $\langle g \rangle$. We use lemma 3.5 and formula 3.7. We have already noted how $W(g)$ acts on $(A^3)_0^g$. On the other hand $W(g)$ acts on the cohomology of a fiber of the resolution as it does on the conjugacy in $\langle g \rangle$. Thus, in our case, the only interesting situation is when $\langle g \rangle$ is either \mathbb{Z}_3 or \mathbb{Z}_4 . In each of these cases we have to look at the \mathbb{Z}_2 representation ϵ , with $\epsilon^2 = 1$ and invariant parts of respective polynomials $(1 + 2\epsilon t + t^2)(1 + (1 + \epsilon)t^2)$ and $(1 + 2\epsilon t + t^2)(1 + (2 + \epsilon)t^2)$. From the resulting polynomials one has to subtract the part related to the fixed points of the action of $W(g)$. The result is as follows. We write the polynomials associated to the respective cyclic groups \mathbb{Z}_2 , \mathbb{Z}_3 and \mathbb{Z}_4 :

`S12(t):=10*((1+t^2)*(1+t^2)-4*(1+t^2));`

`S13(t):=((t^4+2*t^3+2*t^2+1)-4*(1+t^2));`

$$S14(t) := 4 * ((2 * t^4 + 2 * t^3 + 3 * t^2 + 1) - 4 * (1 + 2 * t^2));$$

Finally, we consider 0-dimensional strata which, again, we compute using McKay correspondence:

$$S0(t) := 4 * (1 + 3 * t^2) + (12 + 4) * (1 + 4 * t^2);$$

Calculating the sum $P(t) := S3(t) + S12(t) + S13(t) + S14(t) + S0(t)$ we get:

Proposition 4.4. *The Poincaré polynomial of a crepant resolution of A^3/S_4 , $X \rightarrow A^3/S_4$, is*

$$P_X(t) = t^6 + 20t^4 + 14t^3 + 20t^2 + 1.$$

5. BUILDING UPON ABELIAN SURFACES: $d = 2$.

In this section we consider A of dimension 2, i.e. an abelian surface.

We will take the group S_r with the standard representation $\rho_{\mathbb{C}}$ and the Kummer construction applied to A will give the series of symplectic manifolds $Kum^{(r-1)}$, as noted in section 2.3. In particular a global symplectic resolution exists and McKay correspondence holds. Therefore we can compute the Poincaré polynomial of a crepant resolution of A^r/S_{r+1} with our method, i.e. using 3.9. Invariants of Beauville's generalized Kummer manifolds have been dealt with by Göttsche [Göt93], Göttsche and Soergel [GS93], Debarre, [Deb99], Sawon, [Saw04] and Nieper-Wißkirchen, [NW04], [NW02].

The first step is computing the Poincaré polynomial of the quotient A^r/S_{r+1} , which is obtained by calculating the invariant parts of the representation on $\Lambda^* H^*(A^r, \mathbb{C})$ which is generated by wedge powers of $4\rho_{\mathbb{C}}$, see 3.2.

Next we are to understand the resolution of singularities of $Y = A^r/S_{r+1}$; for this purpose we split this quotient into strata related to points with a fixed isotropy group. We recall some standard facts and definitions regarding the group of permutations, see e.g. [JK81]:

- the conjugacy classes of elements in S_n are determined by their decomposition into cycles and are described by partitions of n , that is sequences of positive integers whose sum is n ,
- $(a_i^{b_i}) = (a_1^{b_1}, \dots, a_m^{b_m})$, where $a_1 > \dots > a_m > 0$ and b_i are positive integers, denotes partition consisting of b_i copies of a_i , so that $b_1 \cdot a_1 + \dots + b_m \cdot a_m = n$,
- for the partition $(a_i^{b_i}) = (a_1^{b_1}, \dots, a_r^{b_r})$ define its length equal to $b_1 + \dots + b_m$, in other words this is the length of the sequence of a_i 's, each of them repeated b_i times,
- the Poincaré polynomial of S_n is, by definition, $P_{S_n}(t) = \sum_0^{n-1} \kappa_i t^{2i}$, where κ_i is the number of partitions of n of length $n - i$,
- we say that partition $(a_i^{b'_i})$ divides (or it is a refinement of) partition $(a_j^{b_j})$ if the sequence of a'_i 's (with repetitions counted by b'_i 's) can be divided into disjoint sequences whose sums yield a_j 's (with repetitions counted by b_j 's)
- given $\sigma \in S_n$, whose decomposition into cycles gives partition $(a_i^{b_i})$, it determines a Young subgroup $S(\sigma) \simeq S_{a_1}^{b_1} \times \dots \times S_{a_m}^{b_m}$, [JK81, Sect. 1.3], the conjugacy class of this group in S_n will be denoted by $S(a_i^{b_i})$.
- $N(S(a_i^{b_i}))/S(a_i^{b_i}) \simeq S_{b_1} \times \dots \times S_{b_m}$, c.f. [JK81, Sect. 4.1]; we denote this group by $W(a_i^{b_i})$.

Fix coordinates (e_0, e_1, \dots, e_r) on A^{r+1} , with $A^r \subset A^{r+1}$ defined by equation $e_0 + e_1 + \dots + e_r = 0$. For a permutation σ such that $[\sigma]_{S_n} = (a_i^{b_i})$, we take its fixed point set $(A^r)^{\sigma}$. Decomposition of σ into cycles gives equations of $(A^r)^{\sigma}$: for example a

cycle $(0, \dots, m)$ yields equations $e_0 = \dots = e_m$. The same is fixed by the respective Young group $S(\sigma)$. On the other hand, each partition defines a closed subset of the quotient $Y = A^r/S_{r+1}$ consisting of orbits of points fixed by the respective conjugacy class of S_{r+1} . Inside this set there is a dense subset consisting of orbits of points whose stabilizer is in the conjugacy class $S(a_i^{b_i})$, we will denote it by $Y(a_i^{b_i}) = Y(a_1^{b_1} \dots a_m^{b_m})$. In particular $Y = \overline{Y(1^{r+1})}$ and the set of fixed points of S_{r+1} is $Y((r+1)^1)$. By $\widehat{Y(a_i^{b_i})}$ we denote the normalization of the closure $\overline{Y(a_i^{b_i})}$.

We have the restriction of the quotient map $(A^r)^\sigma \rightarrow \widehat{Y(a_i^{b_i})} \rightarrow \overline{Y(a_i^{b_i})}$.

Sets $Y(a_1^{b_1} \dots a_m^{b_m})$ determine a stratification of both Y and its resolution $X \rightarrow Y$, the inverse image of $Y(a_1^{b_1} \dots a_m^{b_m})$ will be denoted by $X(a_1^{b_1} \dots a_m^{b_m})$. Below, we list the facts regarding these sets needed to compute the cohomology of X .

Lemma 5.1. *In the above set up the following holds, with $[\sigma]_{S_n} = (a_i^{b_i})$:*

- (1) *the number of fixed points, i.e. $\#Y((r+1)^1)$, is equal $(r+1)^4$ and, more generally, the number of components of $Y(a_i^{b_i})$ is $(\text{GCD}(a_i))^4$, where GCD stands for greatest common divisor,*
- (2) *the sets $(A^r)^\sigma$ and $Y(a_i^{b_i})$ are of pure dimension $2(l-1)$, where l is length of $(a_i^{b_i})$,*
- (3) *the set $Y(a_i^{b_i})$ is contained in the closure $\overline{Y(a_j^{b_j})}$ if and only if partition $(a_j^{b_j})$ divides $(a_i^{b_i})$,*
- (4) *$\overline{(A^r)_0^\sigma} = (A^r)^\sigma$ and the morphism $(A^r)^\sigma \rightarrow \widehat{Y(a_i^{b_i})}$ is quotient by $W(a_i^{b_i})$,*
- (5) *the resolution $f : X \rightarrow Y$ is locally product as in definition 3.4 and we have the following version of 3.5*
 - *the map $X(a_1^{b_1} \dots a_m^{b_m}) \rightarrow Y(a_1^{b_1} \dots a_m^{b_m})$ is étale fiber bundle whose fiber $F(a_1^{b_1} \dots a_m^{b_m})$ is isomorphic to the product $F(a_1)^{\times b_1} \times \dots \times F(a_m)^{\times b_m}$ and has Poincaré polynomial equal to $P_{S_{a_1}}^{b_1} \dots P_{S_{a_m}}^{b_m}$*
 - *the action of $W(a_i^{b_i})$ lifts to the product $(A^r)^\sigma \times F(a_i^{b_i})$ with $W(a_i^{b_i}) \simeq S_{b_1} \times \dots \times S_{b_m}$ acting on $F(a_i^{b_i}) \simeq F(a_1)^{\times b_1} \times \dots \times F(a_m)^{\times b_m}$ by permuting respective factors of the product, that is S_{b_i} permuting factors of $F(a_i)^{\times b_i}$,*
 - *there is a commutative diagram*

$$\begin{array}{ccccc}
 (A^r)^\sigma \times F(a_i^{b_i}) & \longrightarrow & ((A^r)^\sigma \times F(a_i^{b_i})) / W(a_i^{b_i}) & \longleftarrow & X(a_i^{b_i}) \\
 \downarrow & & \downarrow & & \downarrow \\
 (A^r)^\sigma & \longrightarrow & \widehat{Y(a_i^{b_i})} & \longleftarrow & Y(a_i^{b_i})
 \end{array}$$

where the horizontal arrows on the left hand side are quotient maps while these on the right hand side are inclusions onto open subsets

Proof. Most of the above claims follow by explicit calculations and the discussion preceding lemma. For example, the set of fixed points of the action of S_{r+1} is defined in A^{r+1} by equations $e_0 = \dots = e_r$ and $e_0 + \dots + e_r = 0$ hence can be identified with these points in A whose $(r+1)$ -th multiple is zero. The cohomology of the special fiber of resolution of A^r/S_{r+1} , that is of $F((r+1)^1)$, is known by

[Kal02]. The case of $F((a_i^{b_i}))$ follows because of the uniqueness result from [FN04]. This yields that the resolution is locally product in the sense of 3.4. \square

The above lemma provides us with a general layout for computing cohomology of a generalized Kummer variety. In the present section we do explicit calculations of the Poincaré polynomial for the generalized Kummer manifolds of dimension 6 which is a resolution of A^3/S_4 .

Again, the following lines in typewriter type are in the form `maxima`, [Sch07].

First, we write the Poincaré polynomials of fibers of the resolution over the respective strata $F211(t) := 1+t^2$; $F31(t) := 1+t^2+t^4$;

$F22(t) := (1+t^2)^2$; $F4(t) := 1+t^2+2*t^4+t^6$;

Next we write the polynomials for the surface A and its quotient A/\mathbb{Z}_2 where \mathbb{Z}_2 acts on A by multiplying by (-1) .

$A(t) := (1+t)^4$; $B(t) := 1+6*t^2+t^4$;

The next line describes cohomology of $(A \times \mathbb{P}^1 \times \mathbb{P}^1)/\mathbb{Z}_2$ where \mathbb{Z}_2 acts on A as above while its action on $\mathbb{P}^1 \times \mathbb{P}^1$ interchanges the factors. In terms of the \mathbb{Z}_2 action the Poincaré polynomial of the product is $(1 + \epsilon \cdot t)^4 \cdot (1 + (1 + \epsilon) \cdot t^2 + t^4)$ where $\epsilon^2 = 1$. Note that we write both, the polynomial of A as well as $F22$ depending on the group action. The invariant part of this action has the following polynomial.

$C(t) := 1+7*t^2+4*t^3+8*t^4+4*t^5+7*t^6+t^8$;

And finally the Poincaré polynomial of A^3/S_4 which we calculate by looking at the invariants of the respective representation, as in [FH91, Part I]:

$Q(t) := 1+6*t^2+4*t^3+22*t^4+24*t^5+62*t^6+$

$24*t^7+22*t^8+4*t^9+6*t^{10}+t^{12}$;

Now we compute virtual Poincaré polynomials of strata of Y and X , denoted by R and S , respectively. The first are the fixed points.

$R4(t) := 4^4$; $S4(t) := R4(t)*F4(t)$;

Next, take an element of S_4 whose decomposition consists of two cycles of length two, for example $\sigma = (01)(23)$. Its fixed point set is given by equations $e_0 = e_1$, $e_2 = e_3$, $e_0 + \dots + e_3 = 0$, hence $2 \cdot (e_0 + e_2) = 0$ which makes 16 copies of A . The group $W(2^2) \simeq \mathbb{Z}_2$ acts on each of the components by involution, we use 5.1.

$R22(t) := 16*B(t)-R4(t)$; $S22(t) := 16*C(t)-R4(t)*(1+t^2+t^4)$;

The next one is easy, as $W(3,1)$ is trivial.

$R31(t) := A(t)-R4(t)$; $S31(t) := R31(t)*F31(t)$;

The fixed point set of $\sigma = (01)$ contains both, the fixed point set of $(01)(23)$ and of (012) , the former one consists of 16 copies of A and is the fixed point set of the action of $W(2,1,1) \simeq \mathbb{Z}_2$, because $(01)(23)$ is contained in the normalizer of (01) . Note that the action of $W(2,1,1)$ on $F(2,1,1)$ is trivial.

$R211(t) := A(t)*(B(t)-16)-R31(t)$; $S211(t) := R211(t)*F211(t)$;

Finally, we write down the general stratum and the Poincaré polynomial of the resolution.

$R1111(t) := Q(t)-(R211(t)+R31(t)+R22(t)+R4(t))$;

$S1111(t) := R1111(t)$; $P(t) := S1111(t)+S211(t)+S31(t)+S22(t)+S4(t)$.

All together these prove the following:

Proposition 5.2. *The Poincaré polynomial of the Beauville's generalized Kummer variety, which is given by a crepant resolution of A^3/S_4 , where A is a two dimensional torus, is :*

$$t^{12} + 7t^{10} + 8t^9 + 51t^8 + 56t^7 + 458t^6 + 56t^5 + 51t^4 + 8t^3 + 7t^2 + 1$$

6. BUILDING UPON A 4-DIMENSIONAL ABELIAN MANIFOLD.

In this section we consider 4-dimensional abelian varieties, A , with the action of a finite group G . However we will prescribe the action of G only on the complex cohomology $H^1(A, \mathbb{C})$. In other words we fix the complex representation $\rho_{\mathbb{C}} : G \rightarrow SL(\mathfrak{a})$ and we do not require that it comes from an integral representation $\rho_{\mathbb{Z}}$.

Moreover we will take the three groups in the theorem 2.7 which have a four dimensional complex representation of type $V \oplus V^*$; by the theorem there exists a local symplectic resolution of $V \oplus V^*/G$ and in the first two cases if $A = S \times S$, where S is an abelian surface, also a global one (see the end of section 2.3).

Our main tools in this section are the semismallness property of symplectic resolution, namely 2.5, and the Lefschetz fixed point formula, namely 2.1.

6.1. The Binary Tetrahedral Group.

Theorem 6.1. *Let $G = Q_8 \rtimes \mathbb{Z}_3$ be the binary tetrahedral group acting on a 4-dimensional complex torus A such that its action on the complex cohomology group $H^1(A, \mathbb{C})$ is equivalent to the representations $S_1 \oplus S_2$, as in 2.7. Then the quotient A/G does not admit a (global) symplectic resolution.*

Proof. We will use the description of G and its representation given in [LS08], see also 2.1.

The following are the non trivial subgroups of T : $T > M_i > H_i$ for $i = 1, \dots, 4$, where H_i are the 4 conjugate 3-Sylow subgroups and $M_j = \langle -1, H_i \rangle$ are four conjugate subgroups of order 6; $Q_8 > L_j$ where L_j are 3 conjugate subgroups of order 4 generated respectively by I, J, K . Moreover it has a nontrivial center equal to $\langle -1 \rangle$ contained in M_i and in L_j . Note that any two of the groups M_j generate G and $\langle -1 \rangle$ is contained in any non-trivial subgroup of G of order different from 3.

The element -1 acts on $H^1(A, \mathbb{C})$ as the diagonal matrix with all eigenvalue -1 while a generator of K_i acts on $H^1(A, \mathbb{C})$ as the diagonal matrix with eigenvalues $\{-1, -1, e_6, e_6^5\}$, where e_6 is a 6-th primitive root of unity. This implies, in particular, that fixed points of -1 , as well as of K_j , are isolated; by the Lefschetz fixed point formula, 2.1, we get that there are 2^8 and, respectively, 2^4 of them.

For each j the set $Fix(M_j)$ is clearly contained in $Fix(-1)$. We claim that, if A/G admits a symplectic resolution, then $Fix(-1)$ is the union of $Fix(M_j)$. Indeed, isolated quotient symplectic singularities have no crepant resolution in dimension > 2 , see (2.5); so any point of $Fix(-1)$ must be contained in $Fix(H_i)$ for some i , since the groups H_i are the only subgroups not containing -1 (and therefore they are the only ones whose fixed points sets are not contained in $Fix(-1)$).

Therefore, if A/G admits a symplectic resolution then we can write $Fix(-1)$ as a disjoint union of four copies of $(Fix(M_j) \setminus Fix(G))$ and of $Fix(G)$ (note that any two $Fix(M_j)$ intersect along $Fix(G)$).

Thus, if s is the cardinality of $Fix(G)$, we get the following equality

$$2^8 = |Fix(-1)| = 4(|Fix(M_j)| - s) + s = 4(2^4) - 3s = 2^6 - 3s$$

which is a contradiction. \square

6.2. The Dihedral group of order 6.

Theorem 6.2. *Let $G = D_6 = S_3$ be acting on a 4-dimensional complex torus A so that its representation on the complex cohomology is the sum of two copies of the standard representation. Suppose that the quotient of the action A/G admits a*

symplectic resolution of singularities, $X \rightarrow A/G$. Then the complex cohomology of X is uniquely determined, that is the Poincaré polynomial is as follows:

$$P_X = 1 + 7t^2 + 8t^3 + 108t^4 + 8t^5 + 7t^6 + t^8.$$

Remark 6.3. In section 5 we have explained how to compute $P_X(t)$ when X is the resolution of $(A^2)^n/S_{n+1}$, A^2 being a 2-dimensional abelian variety; moreover we have done all the computations for the case S_4 . One can compute the case of S_3 in this way and it will turn out the same formula. So even if one can construct a different resolution of A/G , this resolution will have the same cohomological type of the 4-dimensional case of Beauville's serie.

Proof. The structure of conjugacy classes of subgroups of D_6 is very simple, namely there are two proper non-trivial classes of the normal subgroup \mathbb{Z}_3 and of three \mathbb{Z}_2 subgroups. By Lefschetz fixed-point formula, 2.1, a generator of \mathbb{Z}_3 have 81 isolated fixed points, which by the lemma 2.5, are fixed points of the whole group. On the other hand any order two element t acts with fixed point being a union of, say m , abelian surfaces. Since their Weyl group is trivial the formula in 3.9 yields the following result

$$P_X = 1 + (6 + m)t^2 + (4 + 4m)t^3 + (102 + 6m)t^4 + (4 + 4m)t^5 + (6 + m)t^6 + t^8.$$

We have only to prove that $m=1$. The argument we use is similar to the one used at the end of 4.2.

Note that all components of $Fix(t)$ are numerically equivalent and $Fix(t)^0$ is a subgroup of A . This is because the action of t is algebraic and they are fibers of the map $A \rightarrow A/Fix(t)^0$ (where $Fix(t)^0$ denotes the zero component of $Fix(t)$). The action of t descends to the two dimensional torus $A/Fix(t)^0$ as an involution with 16 fixed points of order two. $Fix(t)$ is a subgroup of A and it descends to a subgroup of the group of points of order 2 in $A/Fix(t)^0$, hence m is a power of 2. On the other hand, let us take another involution t' in D_6 then $Fix(D_6) = Fix(t) \cap Fix(t')$ and since they intersect transversally we get

$$81 = |Fix(D_6)| = m(Fix(t)^0) \cdot m(Fix(t')^0).$$

Thus m is also a power of 3; all this implies that $m = 1$. □

6.3. The Dihedral group of order 8.

Theorem 6.4. *Let $G = D_8 = (\mathbb{Z}_2^2) \rtimes \mathbb{Z}_2$ be acting on a 4-dimensional complex torus A so that its representation on the complex cohomology is the sum of two copies of the standard representation of D_8 as motions of a square. Suppose that the quotient of the action A/G admits a symplectic resolution of singularities, $X \rightarrow A/G$. Then the complex cohomology of X is uniquely determined, that is the Poincaré polynomial is as follows:*

$$P_X(t) = t^8 + 23t^6 + 276t^4 + 23t^2 + 1.$$

Remark 6.5. Similarly to the previous case note that if $A = S \times S$, with S an abelian surface, the symplectic resolution exists, see 2.3. The $P_X(t)$ we obtain is therefore the Poincaré polynomial of this manifold, other possible resolutions will have the same polynomial.

Proof. Let us recall some trivial fact regarding the standard representation of D_8 and its conjugacy classes. The center of the group consists of $\langle -1 \rangle$. There are two

conjugate elements of order 4, namely $\pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and two classes of elements of order 2, namely $\pm A := \pm \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\pm B := \pm \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

From Lefschetz fixed point formula, 2.1, we compute that the order 4 elements have $2^4 = 16$ fixed points, which, by the above lemma 2.5 are the fixed points of the whole group; on the other hand $-I$ has $2^8 = 256$ fixed points, including the previous 2^4 . The other elements of order two have fixed points set being union of abelian surfaces; let us assume that $Fix(\pm A)$ has a components and $Fix(\pm B)$ has b components.

Again, by the lemma 2.5 the points $Fix(-I) \setminus Fix(G)$ are divided among $Fix(\pm A)$ and $Fix(\pm B)$. On the other hand the normalizer of each of the order 2 element (different from $-I$) is a group generated by the element itself and $-I$. Therefore the Weyl group acts on each 2-dimensional component of such an element by involution, with 16 points fixed. Thus we can count the points in the quotient whose isotropy group is non-cyclic and different from the whole group G to get the following identity $|Fix(-I)| - |Fix(G)| = (a + b) \cdot 16 - 2 \cdot |Fix(G)|$, hence $a + b = 17$.

On the other hand, as in the proof of the previous theorem, the numbers a and b are powers of 2, therefore $a = 1$ and $b = 16$ (up to choice of the conjugacy classes). Note that, as in the proof of the previous theorem, we could compute the intersection of the connected components of each of the fixed point sets for both conjugacy classes. For instance we get $ab \cdot Fix(A)^0 \cdot Fix(B)^0 = |Fix(G)| = 16$, hence the intersection is equal to 1.

Let us then compute the cohomology: the Poincaré polynomial of the 4-dimensional stratum, $X([1])$, is

$$(1 + 6t^2 + 22t^4 + 6t^6 + t^8) - 17(1 + 6t^2 + t^4 - 16) - 136;$$

the polynomial of the 3-dimensional strata is

$$17(1 + 6t^2 + t^4 - 16)(1 + t^2)$$

and the one of the 2-dimensional strata is

$$120(1 + 2t^2 + t^4) + 16(1 + 2t^2 + 2t^4).$$

The Poincaré polynomial of X is therefore

$$(1 + 6t^2 + 22t^4 + 6t^6 + t^8) - 17(1 + 6t^2 + t^4 - 16) - 136 + 17(1 + 6t^2 + t^4 - 16)(1 + t^2) + 120(1 + 2t^2 + t^4) + 16(1 + 2t^2 + 2t^4)$$

which, after the simplification, becomes

$$t^8 + 23t^6 + 276t^4 + 23t^2 + 1$$

□

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